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Amplitude-phase modulated composite waves in coupled nonlocal system with self-induced PT symmetric potential

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Abstract: We consider a coupled nonlocal nonlinear Schrödinger equation (nNLSE) with self-induced parity-time (PT) symmetric potential and investigate abundant amplitude-phase modulated composite waves manifesting diverse evolution patterns. It is found that the coupled nonlocal model can be decoupled into nNLSEs with self-induced PT symmetric potential under certain constraints through a general linear transformation with amplitude and phase modulation. Based on the exact solutions of the nNLSEs with self-induced PT potential, we study various composite waves superposed by bright and/or dark soliton solutions, rogue waves, bright/dark soliton and periodic soliton, and present the abundant evolution patterns under amplitude-phase modulation. The results here only demonstrate the characteristics of limited superposed composite waves. In fact, there exist infinite possible evolution patterns of composite waves due to the arbitrary amplitude-phase modulation in coupled nonlocal nonlinear system with self-induced PT symmetric potential.

1. Introduction

In the past few years, the accurate integrable local models such as nonlinear Schrödinger equation (NLSE) and Korteweg–de Vries equation play important roles in the study of the dynamics of localized waves [1]. Based on these models, many exact soliton solutions have been found in the fields of nonlinear fiber optics, hydrodynamics and Bose-Einstein Condensates [2–4]. In recent years, the nonlocal nonlinear Schrödinger equation (nNLSE) was firstly proposed by Ablowitz and Musslimani [5] and has attracted much attention from scholars at home and abroad [8–13]. The nNLSE is in the form of

$$iq_t(x,t) = q_{xx}(x,t) \pm 2q(x,t)q^*(-x,t)q(x,t), \tag{1}$$

where * stands for complex conjugation. Equation (1) is integrable and possesses a linear pair formulation and an infinite number of conservation laws [5]. When $x \to -x$, $t \to -t$ and taking a complex conjugate in Eq. (1), the equation remains unchanged, which means that it is parity-time (PT) symmetric. Indeed, Eq. (1) can be rewritten as a linear form of $iq_t(x,t) = q_{xx}(x,t) \pm 2V(x,t)q(x,t)$, where $V(x,t) = q(x,t)q^*(-x,t)$ is a self-induced nonlinear potential and satisfies the PT symmetric condition, i.e. $V(x,t) = V^*(-x,t)$ [5]. However, its PT symmetry will be broken by the self-induced potential for any shift on the center of the solution evolution [6]. It is worth noting that the solution evolution at current coordinate x depends on both the current location x and the opposite location $-x$, which indicates a coupled relationship between the soliton states at the positions of x and $-x$, reminding of quantum entanglement between pairs of particles in quantum system [7,8]. Even though Eq. (1) is related to an unconventional magnetic system [9], such a nonlocal model cannot be directly realized for concrete physical settings [10]. However, the study on the nonlocal nonlinear model has clear physical motivation

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and significance [10]. The physical motivation originates from the relationship between it and its realistic local counterpart based on the fact that Eq. (1) can be mapped to the local ones that govern wave propagation in a variety of real physical systems. So far, Eq. (1) has been studied extensively by inverse scattering transform [5], Darboux transformation [11–14], bilinear method [15], or direct reduction approach [16] etc., and diverse soliton solutions have been obtained such as breathing one-soliton solution [5], rogue waves with singular peaks [11], singular solutions [12], breather solutions [13] and N-soliton solutions [14] etc. Especially, in contrast with standard local NLSE, Eq. (1) admits the simultaneous existence of bright and dark soliton solutions under the same sign of dispersion [6].

Following the integrable PT symmetric nNLSE (1), the coupled nNLSE with nonlocal self-phase modulation (nSPM), cross-phase modulation (nXPM) and four-wave mixing (nFWM) was paid extensive attention [17–19]. In the absence of nFWM, the coupled nNLSE can be reduced to nonlocal Manakov model, and Bright-Bright, Dark-Dark and Bright-Dark solitons etc. have been reported [17,18] and these solutions exhibit distinct wave structures due to the nonlocality. With the effect of nFWM, special soliton solutions including Kuznetsov-Ma breather, Akhmediev breather, and Peregrine soliton, soliton (breather) lattice were also demonstrated [19]. It is worth mentioning that the conversion between Bright-Dark solitons and the breathers can happen under the effect of nFWM [19]. It is well known that the FWM arises from the nonlinear response of bound electrons in the medium to electromagnetic field [2], which can be expressed in the forms of $q_2(x,t)q_1^*(x,t)q_2(x,t)$, $q_1(x,t)q_2^*(x,t)q_1(x,t)$ and so on under the phase matching [2,19–21]. As an important nonlinear effect, the FWM can be applied to optical sampling, pulse generation, wavelength conversion etc. [22]. Up to date, the research on the coupled nonlocal systems with self-induced PT symmetric potential, especially including nFWM effect, is far insufficient. And the evolution patterns of diverse composite waves in nonlocal systems have not been explored yet.

Motivated by the above works, here we focus on the coupled nNLSE including nSPM, nXPM and nFWM and investigate abundant amplitude-phase modulated composite waves manifesting diverse evolution patterns. With the help of an introduced general linear transformation and the exact solutions of the nNLSE with self-induced PT potential, we study various composite waves superposed by bright and/or dark soliton solutions, rogue waves, bright/dark soliton and periodic solitons, and present the abundant evolution patterns under amplitude-phase modulation. Here, the introduced linear transformation includes the information of amplitude and phase, which implies one can achieve superposition of various amplitude-phase modulated solitons. It is essentially different from the linear transformation mentioned in the previous Refs. [20,21].

The paper is organized as follows: in Sec. 2, we decouple the coupled nonlocal model into two independent nNLSEs with self-induced PT symmetric potential through introducing a general linear transformation with amplitude-phase modulation; in Sec. 3, we present the exotic patterns of composite waves superposed by bright and/or dark soliton, rogue waves, bright/dark soliton and periodic soliton, and explore the effect of amplitude-phase modulation on the evolution patterns of a family of composite waves; in Sec. 4, the conclusion is summarized.

2. General linear transformation with amplitude-phase modulation

Here we consider a generalized coupled nonlocal NLSE involving nSPM, nXPM and nFWM [18,19]:

$$iq_{1t}(x,t) + \sigma q_{1xx}(x,t) + [aq_1(x,t)q_1^*(-x,t) + cq_2(x,t)q_2^*(-x,t)]q_1(x,t) + dq_2^2(x,t)q_1^*(-x,t) = 0, \tag{2a}$$

$$iq_{2t}(x,t) + \sigma q_{2xx}(x,t) + [cq_1(x,t)q_1^*(-x,t) + aq_2(x,t)q_2^*(-x,t)]q_2(x,t) + dq_1^2(x,t)q_2^*(-x,t) = 0, \tag{2b}$$

where $q_{1,2}(x,t)$ represent complex fields of the variables x and t, σ is the coefficient of group velocity dispersion (GVD), a, c and d denote the nSPM, nXPM and the nFWM effects,

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respectively. The last terms in Eqs. (2a) and (2b) are responsible for nonlocal coherent coupling effect which is a special case of a more general FWM process [23]. It should be pointed out that Eq. (2) can be used to describe the propagation of nonlinear waves in weakly anisotropic or weakly birefringent medium since the term of degree of birefringence is not included. The degree of birefringence, closely related to the group-velocity mismatch and walk-off effect, can be neglected in the case of weak birefringence [23]. Oppositely, in the case of high birefringence, the degree of birefringence and the walk-off effect resulting from group velocity mismatch must be included in the theoretical model. Clearly, the coupled nonlocal system (2) includes a self-induced nonlinear PT-symmetric potential for the single equation [5], i.e.$V_j(x,t) = aq_j(x,t)q_j^*(-x,t) + cq_{j\pm1}(x,t)q_{j\pm1}^*(-x,t)$, ($j = 1, 2$ corresponding '+' and '-', respectively) that satisfies the PT-symmetric condition $V_j(x,t) = V_j^*(-x,t)$. It is noted that through the variable transformations $x = i\hat{x}$, $t = -\hat{t}$, $q_{1,2}(x,t) = \hat{u}_{1,2}(\hat{x},\hat{t})$ [24], Eq. (2) can be converted to its local counterpart, i.e.

$$i\hat{u}_{j\hat{t}}(\hat{x},\hat{t}) + \sigma\hat{u}_{j\hat{x}\hat{x}}(\hat{x},\hat{t}) - [a\hat{u}_j(\hat{x},\hat{t})\hat{u}_j^*(\hat{x},\hat{t}) + c\hat{u}_{j\pm1}(\hat{x},\hat{t})\hat{u}_{j\pm1}^*(\hat{x},\hat{t})]\hat{u}_j(\hat{x},\hat{t}) - d\hat{u}_{j\pm1}^2(\hat{x},\hat{t})\hat{u}_j^*(\hat{x},\hat{t}) = 0, \quad (3)$$

($j = 1, 2$, corresponding '+' and '-', respectively), which govern the dynamics of two coherently coupled waves in isotropic nonlinear medium [25–27]. Similarly, Eq. (2) can be recovered through reverse variable transformations $\hat{x} = -ix$, $\hat{t} = -t$, $\hat{u}_{1,2}(\hat{x},\hat{t}) = q_{1,2}(x,t)$ that describe the nonlinear wave propagation in real physical systems [10]. This implies that studying on the nonlocal model (2) has clear physical significance.

To seek the phase-amplitude modulated solutions of Eq. (2), we introduce a general linear transformation of nonlinear wave superposition

$$q_1(x,t) = A_1e^{i\theta_1}\phi_1(x,t) + A_3e^{i\theta_3}\phi_2(x,t), \quad (4a)$$

$$q_2(x,t) = A_2e^{i\theta_2}\phi_1(x,t) - A_4e^{i\theta_4}\phi_2(x,t), \quad (4b)$$

where $\phi_{1,2}$ is arbitrary function of x and t, and the coefficients A_j ($j = 1, 2, 3, 4$) is any non-zero real constant, and θ_j ($j = 1, 2, 3, 4$) ranges from 0 to 2π. That is to say, amplitude-phase modulation in the linear transformation (4) is arbitrary, which implies that the linear transformation (4) covers the linear transformations in Refs. [20,21]. Namely, the linear transformation (4) is much more general and it could theoretically be extended to the other systems. Inserting the linear transformation (4) into Eq. (2) and after a series of complicated calculations (See Appendix B for details), it is found that under the certain constraints $a = d = c/2$ (indicating the positive coherent coupling [28]) $A_2 = e^{il_1\pi}A_1$, $A_4 = e^{il_2\pi}A_3$, $\theta_2 - \theta_1 = k_1\pi$, $\theta_4 - \theta_3 = k_2\pi$, where $l_{1,2}$ and $k_{1,2}$ are integers, Eq. (2) can be decoupled into

$$i\phi_{jt}(x,t) + \sigma\phi_{jxx}(x,t) + \gamma_j\phi_j(x,t)\phi_j^*(-x,t)\phi_j(x,t) = 0, (j = 1, 2) \quad (5)$$

where $\gamma_1 = 2cA_1^2$ and $\gamma_2 = 2cA_3^2$. Equation (5) is a nNLSE with self-induced PT potential and supports numerous exact soliton solutions [11–16,29–32], such as bright, dark solitons [6], rogue waves [11], periodic solitons [13] etc. We will show them in Appendix A in detail for convenience. Thus, through linear transformation (4), rich composite waves of Eq. (2) can be obtained with the aid of diverse soliton solutions of Eq. (5). It should be pointed out that the linear transformation (4) is much more general than the ones reported in Refs. [20,21] (in which all coefficients of $\phi_j(x,t)$ are fixed to special real constants). In fact, the real coefficients A_j and the phases θ_j in the linear transformation (4) will give rise to superposition of various amplitude-phase modulated solitons of Eq. (5), which implies one can achieve the kaleidoscopic complex soliton solutions of Eq. (2) by the jointly modulated amplitude-phase solutions of Eq. (5).

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According to the above constraints, the linear transformation (4) can be further rewritten as

$$q_1(x,t) = A_1 e^{i\theta_1}\phi_1(x,t) + A_3 e^{i\theta_3}\phi_2(x,t), \tag{6a}$$

$$q_2(x,t) = A_1 e^{i(\theta_1+p_1\pi)}\phi_1(x,t) - A_3 e^{i(\theta_3+p_2\pi)}\phi_2(x,t), \tag{6b}$$

where $p_j = l_j + k_j$ $(j = 1, 2)$. It is easy to see that different integer p_j leads to different superposed solution $q_2(x,t)$ even though for the same solution $\phi_j(x,t)$ of Eq. (5), which means that there exist various kinds of linear transformations composed of the same (6a) and different (6b). According to different values of p_j, Eq. (6b) can be divided into four cases: Case i) when p_1 and p_2 are even and odd numbers, respectively, Eq. (6b) reads $q_2(x,t) = A_1e^{i\theta_1}\phi_1(x,t) + A_3e^{i\theta_3}\phi_2(x,t)$, that is $q_1(x,t) = q_2(x,t)$, i.e. q_1 and q_2 are equal in amplitude and in phase; Case ii) when p_1 and p_2 are odd and even numbers, respectively, then $q_2(x,t) = -A_1e^{i\theta_1}\phi_1(x,t) - A_3e^{i\theta_3}\phi_2(x,t)$, that is $q_1(x,t) = -q_2(x,t)$, namely, q_1 and q_2 are equal in amplitude but out of phase; Case iii) when p_1 and p_2 are odd numbers, then $q_2(x,t) = -A_1e^{i\theta_1}\phi_1(x,t) + A_3e^{i\theta_3}\phi_2(x,t)$; Case iv) when p_1 and p_2 are even numbers, then $q_2(x,t) = A_1e^{i\theta_1}\phi_1(x,t) - A_3e^{i\theta_3}\phi_2(x,t)$. In combination with Eq. (6a), for Case iii) and Case iv), the two components q_1 and q_2 have different amplitudes and phases, moreover, their amplitudes and phases can be freely modulated since A_1, A_3, θ_1 and θ_3 are arbitrary. This leads to a series of different linear transformations that can achieve diverse amplitude-phase modulated composite waves of coupled nNLSE (2) with PT symmetric potential, which is much more general than the ones with fixed coefficients for local counterparts [20,21]. Such amplitude-phase modulated composite waves in coupled nonlocal system with PT symmetric potential have never been studied in literatures. In order to demonstrate the amplitude-phase modulation, in this paper we will mainly explore the linear transformation described by Eq. (6a) and Case iv. In this case, in order to show the amplitude-phase modulation more clearly, we rewrite the linear transformation as

$$q_j(x,t) = (A_1e^{i\theta_1}, (-1)^{j+1}A_3e^{i\theta_3})\begin{pmatrix}\phi_1(x,t)\\ \phi_2(x,t)\end{pmatrix}, j = 1, 2. \tag{7}$$

It is worth noting that the values of A_1, A_3, θ_1 and θ_3 are arbitrary, which indicates that q_1 and q_2 are the composite waves of ϕ_1 and ϕ_2 with *arbitrary* amplitude-phase modulation. In other words, different amplitude-phase modulation of ϕ_1 and ϕ_2 follows different linear transformation, which results in various composite waves of Eq. (2). Table 1 lists several modulation parameters $e^{i\theta_j}$ corresponding to special values of θ_j. The modulation parameter space for linear transformations (7) is shown by the all plane in Fig. 1, where the special points A-H respectively correspond to the linear transformations with the modulation parameters listed in Table 1. For example, A and E correspond to the linear transformation of $\theta_1 = b_1\pi$,$\theta_3 = b_2\pi$ with b_1 and b_2 being integers,

$$q_j(x,t) = \pm(A_1, (-1)^{j+1}A_3)\begin{pmatrix}\phi_1(x,t)\\ \phi_2(x,t)\end{pmatrix}, j = 1, 2 \tag{8}$$

Table 1. The phase θ_j and the corresponding modulation parameters $e^{i\theta_j}$

θ_j	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$
$A_je^{i\theta_j}$	A_j	$A_j\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$	iA_j	$A_j\left(i\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$	$-A_j$	$-A_j\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$	$-iA_j$	$A_j\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

Mareeswaran et al. and Jia et al. have obtained and investigated nonlinear localized composite waves of coherently coupled NLSE through linear transformation (8) with '+' and fixed parameters $A_1 = A_3 = 1/2$ or $A_1 = A_3 = 1/\sqrt{2}$ [20,21]. It should be pointed out that the linear transformation

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(8) is one special case of the linear transformation (7) which is much more general than the reported ones [20,21].

With the help of linear transformation (7) that contains information of arbitrary amplitude and phase, one can obtain a family of composite waves consisting of the superposition of two exact solutions $\phi_{1,2}$ of Eq. (5) that admits exact bright, dark [6], rogue wave [11] and lattice solitons [13] etc. We will discuss the interesting composite waves with amplitude-phase modulation in the coupled nonlocal PT symmetric system (2) based on the general linear transformation (7) as below. For brevity, we will define the new composite waves superposed of two nonlinear waves X and Y as X-Y. For example, BS-DS, BS-BS and DS-DS represent the superposition of bright-dark soliton (BS-DS), bright-bright soliton (BS-BS) and dark-dark soliton (DS-DS), respectively.

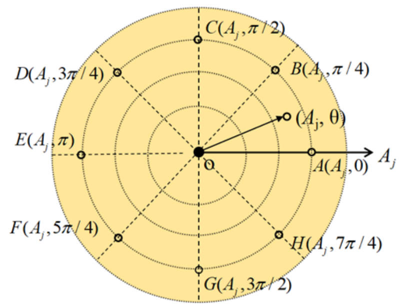

Fig. 1. The modulation parameter space corresponding to the modulation parameters listed in Table 1 for linear transformations (7)

3. Composite waves based on amplitude-phase modulation

3.1. Composite bright and/or dark waves

In contrast to the local NLSE that supports either bright or dark soliton [5], Eq. (5) admits simultaneously both the bright and dark soliton solutions [Eq. (10) and Eq. (11)] under the same constraints [6]. This allows us to discuss the composite waves superposed by bright and dark solitons in coupled nonlocal PT symmetric system (2). Following the linear transformation (7), the amplitude of BS-DS can be expressed as

$$|q_{1,2}| = [A_1^2|\phi_1(x,t)|^2 + A_3^2|\phi_2(x,t)|^2 \pm 2A_1A_3|\phi_1(x,t)||\phi_2(x,t)|\cos(\Delta_\theta + \varphi_1 - \varphi_2)]^{1/2}, \quad (9)$$

where '$\pm$'correspond to $q_{1,2}$, $|\phi_1(x,t)| = |\eta_1 \operatorname{sech}(\eta_1 x)|$ and $|\phi_2(x,t)| = |\mu_2 \tanh(\mu_2 x)|$ are respectively the amplitudes of bright [Eq. (10)] and dark [Eq. (11)] soliton solutions, $\varphi_1 = {\eta_1}^2 t/2$ and $\varphi_2 = -\mu_2^2 t$ are respectively the phases of ϕ_1 and ϕ_2, where η_1, μ_2 are the amplitudes of bright and dark solitons, respectively, and $\Delta_\theta = \theta_1 - \theta_3$ denotes the relative phase between the bright and dark solitons. As indicated above, $A_{1,3}$ and $\theta_{1,3}$ are arbitrary, hence the amplitude (9) of BS-DS can be modulated by the amplitudes and phases. Figure 2 shows the evolutions of the BS-DS under different phase-modulation Δ_θ at a given equal amplitude parameters $A_1 = A_3$. It is clear to see that the BS-DS appears alternating patterns of bright and dark spots, and the components q_1 and q_2 have the complementary characteristics, which results from the conservation of energy. With the increase of the relative phase Δ_θ, both bright and dark spots located at t = 0 move along the negative t direction till $\Delta_\theta = \pi$, and the bright (dark) spots at t = 0 gradually evolve into the dark (bright) ones, as shown in Figs. 2(a) to 2(d).

Figures 3(a) and 3(b) depict the evolution of BS-DS at a fixed relative phase $\Delta_\theta = \pi/4$. For different amplitude parameters η_1 and μ_2, BS-DS behaves bright or dark soliton appearing alternately the ellipse-shaped spots on two sides of the axis of x = 0. And the ratio of η_1 to μ_2 is

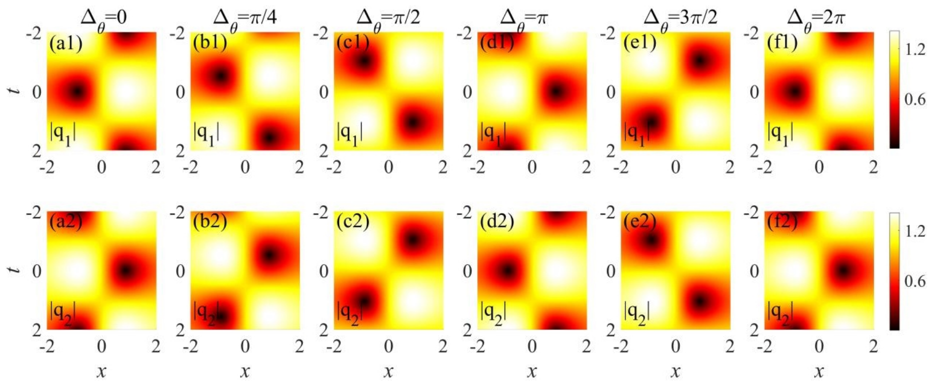

Fig. 2. Composite waves BS-DS with different phase modulation at a given equal amplitude parameters $A_1 = A_3 = 1$: (a) $\Delta_\theta = 0$; (b) $\Delta_\theta = \pi/4$; (c) $\Delta_\theta = \pi/2$; (d) $\Delta_\theta = \pi$; (e) $\Delta_\theta = 3\pi/2$; (f) $\Delta_\theta = 2\pi$; the other parameters are $\eta_1 = 1$, $\mu_2 = 1$,respectively.

larger than 1, the bright soliton dominates the evolution of the composite wave [see Fig. 3(a)], while the ratio is smaller the dark soliton dominates its revolution [see Fig. 3(b)]. In addition, by changing the amplitudes A_1 and A_3 of the bright and dark solitons can achieve the conversion of the dominated role as well.

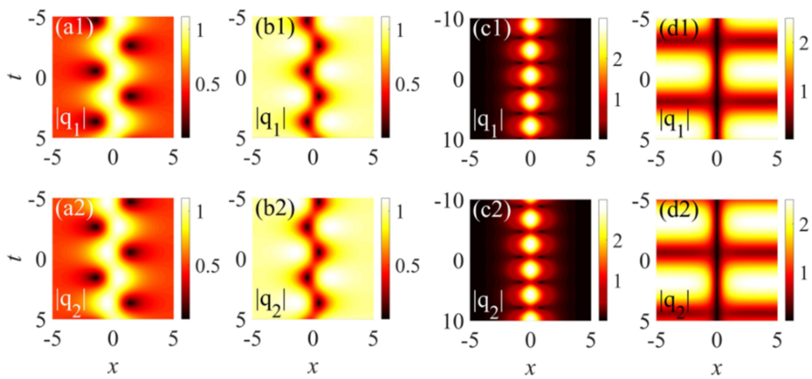

Fig. 3. Composite waves of (a-b) BS-DS, (c) BS-BS, (d) DS-DS with a fixed relative phase $\Delta_\theta = \pi/4$ and equal amplitudes $A_1 = A_3 = 1$. Other parameters are (a) $\eta_1 = 3$, $\mu_2 = 1$; (b) $\eta_1 = 1$, $\mu_2 = 1.5$; (c) $\eta_1 = 2$, $\eta_2 = 1$; (d) $\mu_1 = 1$, $\mu_2 = 1.5$.

In fact, in accordance with the expression (9) of amplitude of $q_{1,2}$, regardless of the forms of the solution $\phi_{1,2}$, the composite wave always evolves periodically except for $\Delta_\theta = \varphi_2(x,t) - \varphi_1(x,t)$, but the period T is $4\pi/({\eta_1}^2 + 4{\mu_2}^2)$, $4\pi/({\eta_1}^2 - {\eta_2}^2)$ and $2\pi/({\mu_2}^2 - {\mu_1}^2)$ for BS-DS, BS-BS ($\eta_1 \neq \eta_2$) and *DS-DS* ($\mu_1 \neq \mu_2$), respectively. Figures 3(c) and 3(d) depict the evolutions of the composite waves of BS-BS and DS-DS, respectively. It can be seen from Fig. 3(c) that BS-BS exhibits a breathing behavior like the Kuznetsov-Ma breather [13] when the condition $\eta_1 \neq \eta_2$ is satisfied. For the DS-DS composite waves, its background periodically evolves with the period $2\pi/({\mu_2}^2 - {\mu_1}^2)$ keeping its dip invariant when $\mu_1 \neq \mu_2$, which is shown in Fig. 3(d). Taking BS-BS as example, Fig. 4 plots the relation of the phase and amplitude of BS-BS wave with the change of relative phase Δ_θ. It is clear to see that for different periods T, with the increase of Δ_θ, the phases of q_1 and q_2 follow the same law but with different magnitudes [see Fig. 4(a1) and 4(a2)], while their amplitudes are opposite [see Fig. 4(b1) and 4(b2)]. Also, as the period T decreases, the amplitudes of q_1 and q_2 increase while the phases decrease.

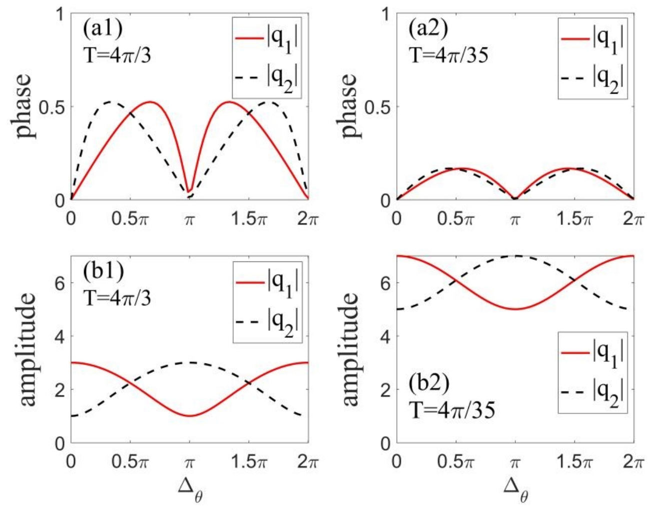

Fig. 4. The relations between relative phase Δ_θ and phase and amplitude of composite wave BS-BS at x = 0. The parameters for (a1) and (b1) are identical to the ones in Fig. 1(c), and the parameters for (a2) and (b2) are $\eta_1 = 6$, $\eta_2 = 1$.

3.2. Composite rogue waves

Rogue waves in coupled nonlocal systems are rarely studied, especially in nonlocal PT symmetric system. In this section, we investigate composite rogue waves superposed by singular (collapsing) rogue waves [Eq. (13)] under amplitude-phase modulation (7), and present some novel patterns and interesting evolution properties. Figure 5 depicts when the relative phase $\Delta_\theta = \theta_1 - \theta_3$ is different, the evolution of the composite rogue wave, named as RW-RW, superposed by two second-order rogue waves with different parameters $s_1 = 0$, $r_1 = -30$ and $s_2 = 0$, $r_2 = 30$, where $s_{1,2}$ and $r_{1,2}$ are free real parameters. It is found that the composite RW-RW can form a hexagon shape consisted of six bright spots corresponding to six collapsed peaks [11]. It can be seen from Fig. 5 that as the relative phase Δ_θ increases, the intensity of the background wave of q_1 changes from strong to weak and then back to strong again, while for the background wave change of q_2 is opposite. Also, the intensity patterns of q_1 and q_2 exchange at $\Delta_\theta = \pi$ [see Figs. 5(c1) and 5(c2)], and both q_1 and q_2 evolve at the period of 2π[see Figs. 5(a1), 5(e1) and 5(a2), 5(e2)].

RW-RW with a certain relative phase $\Delta_\theta = \pi/4$. It is clear to see that by changing parameters of the rogue wave, the collapsed position of the composite RW-RW can be regulated and various patterns such as triangular, rectangular, square and hexagon are formed, as shown in Figs. 6(a) to 6(d), respectively. It is also noted that in Figs. 6(a), 6(b) and 6(d), the two components q_1 and q_2 of the composite RW-RW occur collapses at six positions, while in Fig. 6(c), they appear collapses at five positions. Because in Fig. 6(c) two singular peaks overlaps.

3.3. Diverse soliton patterns

The diversity of composite waves of Eq. (2) comes not only from the abundance of amplitude-phase modulation parameters, but also from the fact that $\phi_{1,2}$ could be any solutions of Eq. (5). Here, based on the simplest bright/dark soliton [Eqs. (10) and (11)] and periodic soliton (which includes some parameters, where α_2 is related to the period, and β_2, v_2, f_2 are intermediate parameters) [Eq. (12)] of Eq. (5), the evolutions of DS-PS and BS-PS are presented in Fig. 7.

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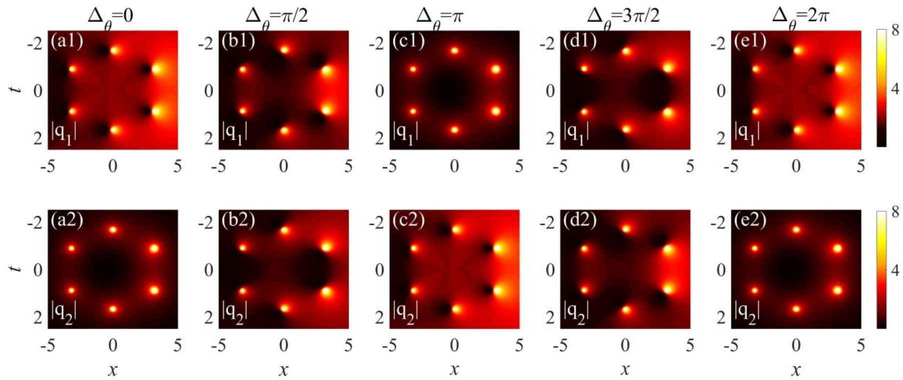

Fig. 5. Composite RW-RW for different amplitude-phase modulation: (a)$\Delta_\theta = 0$;(b)$\Delta_\theta = \pi/2$; (c) $\Delta_\theta = \pi$; (d) $\Delta_\theta = 3\pi/2$; (e) $\Delta_\theta = 2\pi$. Other parameters are $s_1 = 0, r_1 = -30, s_2 = 0, r_2 = 30, A_1 = A_3 = 1$.

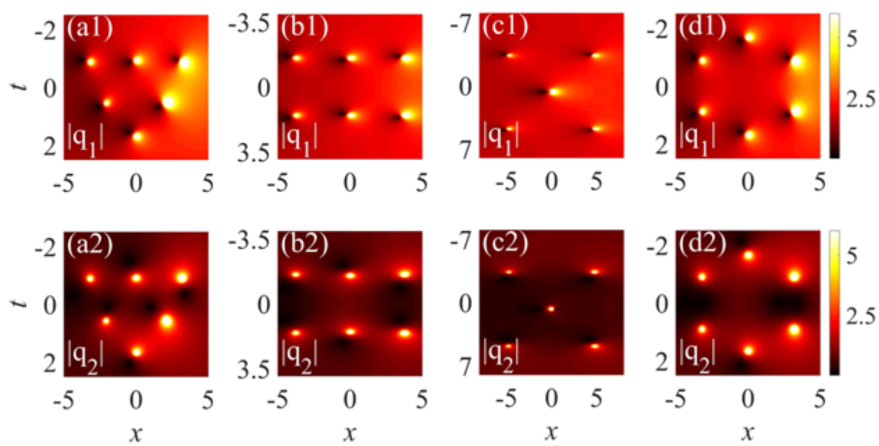

Fig. 6. Composite waves RW-RW with diverse patterns for a fixed relative phase $\Delta_\theta = \pi/4$ and equal amplitudes $A_1 = A_3 = 1$: (a)$s_1 = 6.9$, $r_1 = -100$, $s_2 = -4$, $r_2 = 100$; (b) $s_1 = 0$, $r_1 = -30$, $s_2 = 0$, $r_2 = 30$; (c)$s_1 = 0$, $r_1 = -7$, $s_2 = 0$, $r_2 = 30$; (d)$s_1 = 1$, $r_1 = -50$, $s_2 = -1$, $r_2 = 50$.

Due to the energy conservation of q_1 and q_2, only the evolution of q_1 is shown. It is clear to see that both BS-PS and DS-PS evolve periodically. Comparing the first column to the second column, or the third column to the fourth column, it is found that with the increase of η_1 or μ_1, the shapes of the spots along x = 0 change significantly, resulting from the stronger amplitudes of bright or dark solitons. Comparing the first column to the third column, or the second column to the fourth column, we find that as α_2 increases, the intensity of the waves with the form of fringes becomes weaker and the period becomes smaller. Meanwhile, the number of bright spots along x = 0 increases as α_2 increases.

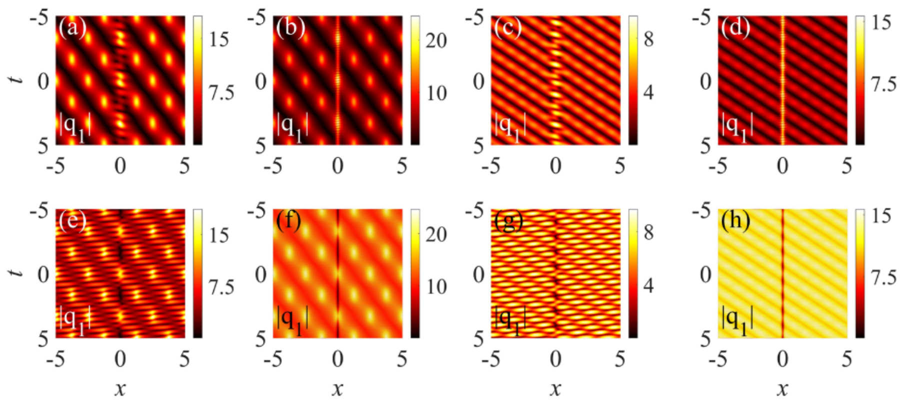

Fig. 7. The component q_1 of diversely modulated soliton patterns of (a)-(d)BS-PS, (e)-(h)DS-PS: (a)$\eta_1 = 4$, $\alpha_2 = 0.7$; (b)$\eta_1 = 10$, $\alpha_2 = 0.7$; (c)$\eta_1 = 4$, $\alpha_2 = 1.2$;(d)$\eta_1 = 10$, $\alpha_2 = 1.2$;(e) $\mu_1 = 4$, $\alpha_2 = 0.7$; (f)$\mu_1 = 10$, $\alpha_2 = 0.7$,(g)$\mu_1 = 4$, $\alpha_2 = 1.2$,(h)$\mu_1 = 10$, $\alpha_2 = 1.2$. Other parameters:$v_2 = \beta_2 = 0, f_2 = 1 - \sqrt{3}/6 \log(2 + \sqrt{3})$.

4. Conclusion and discussion

In this paper, we considered a generalized coupled nNLSE with nSPM, nXPM and nFWM, which shows the physical significance by mapping nonlocal system into local counterpart. With the help of an introduced general linear transformation, the coupled nonlocal model can be decoupled into two nNLSEs with self-induced PT symmetric potential under appropriate constraints. As the linear transformation includes abundant information of amplitude and phase modulation, diverse evolution patterns of the composite waves in coupled PT-symmetric nonlocal system can be achieved. Several composite waves, superposed by bright and/or dark soliton solutions, or two rogue waves, or bright/dark soliton and periodic breather soliton, are exhibited as examples. Especially, the exotic evolution patterns of composite wave RW-RW in nonlocal system are also demonstrated by the superposition of rogue waves under amplitude-phase modulation. The results presented here are difficult to achieve in the local system.

Appendix A: explicit exact solutions of nNLSE (5)

1. Bright and dark soliton [6] and periodic soliton [13] in anomalous GVD focusing nNLSE with $\gamma = 1, \sigma = 1/2$, which are respectively in the form of

$$\phi_j = \eta_j \operatorname{sech}(\eta_j x) e^{i\eta_j{}^2 t/2}, j = 1, 2; \tag{10}$$

$$\phi_j = \mu_j \tanh(\mu_j x) e^{-i\mu_j{}^2 t}, j = 1, 2; \tag{11}$$

$$\phi_j = \frac{\cos(\beta_j)\cosh(B_1 + 2\alpha_j) + \sinh(\alpha_j)\sinh(A_1 - 2i\beta_j)}{\cos(\beta_j)\cosh(B_1) - \sinh(\alpha_j)\sinh(A_1)} e^{it/2}, j = 1, 2, \tag{12}$$

with $A_1 = 2\sinh(\alpha_j)(f_j - ix)$, $B_1 = -2i\sinh(\alpha_j)[\cosh(\alpha_j)t + v_j]$.

Here η_j and μ_j in Eqs. (10) and (11) are related to bright and dark soliton amplitude, respectively, and α_j, β_j, v_j, f_j in Eq. (12) are the same as the corresponding parameters in Ref. [13].

2. Rogue wave in normal GVD focusing nNLSE $\gamma = -2$, $\sigma = -1$ [11]:

$$\phi_j = e^{-2it}\left[1 + \frac{3(2x - 4i\tilde{t} + 1)^2}{4(x - 2i\tilde{t})^3 - 3(x - 6i\tilde{t} + 2i\tilde{s}_1)}\right], j = 1, 2, \tag{13}$$

where $\tilde{s}_1 = r_j - s_j$, $\tilde{t} = t - s_j/2$, $\tilde{s}_1$ is a single nonreducible real parameter after the parameter s_j is removed by time translation, r_j is a free real parameter.

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Appendix B: decoupling process

Inserting the linear transformation (4) into Eq. (2), one can obtain the following equations

$$\begin{aligned}
&iA_1e^{i\theta_1}\phi_{1t}(x,t) + iA_3e^{i\theta_3}\phi_{2t}(x,t) + \sigma A_1e^{i\theta_1}\phi_{1xx}(x,t) + \sigma A_3e^{i\theta_3}\phi_{2xx}(x,t) \\
&+B_1\phi_1^2(x,t)\phi_1^*(-x,t) + B_3\phi_2^2(x,t)\phi_2^*(-x,t) + C_1\phi_1(x,t)\phi_2(x,t)\phi_1^*(-x,t) \\
&+D_1\phi_1(x,t)\phi_2(x,t)\phi_2^*(-x,t) + E_1\phi_2^2(x,t)\phi_1^*(-x,t) + F_1\phi_1^2(x,t)\phi_2^*(-x,t) = 0,
\end{aligned} \tag{14a}$$

$$\begin{aligned}
&iA_1e^{i\theta_1}\phi_{1t}(x,t) - iA_3e^{i\theta_3}\phi_{2t}(x,t) + \sigma A_1e^{i\theta_1}\phi_{1xx}(x,t) - \sigma A_3e^{i\theta_3}\phi_{2xx}(x,t) + \\
&B_2\phi_1^2(x,t)\phi_1^*(-x,t) - B_4\phi_2^2(x,t)\phi_2^*(-x,t) - C_2\phi_1(x,t)\phi_2(x,t)\phi_1^*(-x,t) + \\
&D_2\phi_1(x,t)\phi_2(x,t)\phi_2^*(-x,t) + E_2\phi_2^2(x,t)\phi_1^*(-x,t) - F_2\phi_1^2(x,t)\phi_2^*(-x,t) = 0,
\end{aligned} \tag{14b}$$

where $B_{1,3} = aA_{1,3}^3e^{i\theta_{1,3}} + cA_{1,3}A_{1,3+1}^2e^{i\theta_{1,3}} + dA_{1,3}A_{1,3+1}^2e^{i(2\theta_{1,3+1}-\theta_{1,3})}$, $B_{2,4} = aA_{2,4}^3e^{i\theta_{2,4}} + cA_{2,4}A_{2,4-1}^2e^{i\theta_{2,4}} + dA_{2,4}A_{2,4-1}^2e^{i(2\theta_{2,4-1}-\theta_{2,4})}$, $C_1 = 2aA_1^2A_3e^{i\theta_3} + cA_2^2A_3e^{i\theta_3} - cA_1A_2A_4e^{i(\theta_1+\theta_4-\theta_2)} - 2dA_1A_2A_4e^{i(\theta_2+\theta_4-\theta_1)}$, $C_2 = 2aA_2^2A_4e^{i\theta_4} + cA_1^2A_4e^{i\theta_4} - cA_1A_2A_3e^{i(\theta_2+\theta_3-\theta_1)} - 2dA_1A_2A_3e^{i(\theta_1+\theta_3-\theta_2)}$, $D_1 = 2aA_1A_3^2e^{i\theta_1} + cA_1A_4^2e^{i\theta_1} - cA_2A_3A_4e^{i(\theta_2+\theta_3-\theta_4)} - 2dA_2A_3A_4e^{i(\theta_2+\theta_4-\theta_3)}$, $D_2 = 2aA_2A_4^2e^{i\theta_2} + cA_2A_3^2e^{i\theta_2} - cA_1A_3A_4e^{i(\theta_1+\theta_4-\theta_3)} - 2dA_1A_3A_4e^{i(\theta_1+\theta_3-\theta_4)}$, $E_1 = aA_1A_3^2e^{i(2\theta_3-\theta_1)} - cA_2A_3A_4e^{i(\theta_3+\theta_4-\theta_2)} + dA_1A_4^2e^{i(2\theta_4-\theta_1)}$, $E_2 = aA_2A_4^2e^{i(2\theta_4-\theta_2)} - cA_1A_3A_4e^{i(\theta_3+\theta_4-\theta_1)} + dA_2A_3^2e^{i(2\theta_3-\theta_2)}$, $F_1 = aA_1^2A_3e^{i(2\theta_1-\theta_3)} - cA_1A_2A_4e^{i(\theta_1+\theta_2-\theta_4)} + dA_2^2A_3e^{i(2\theta_2-\theta_3)}$, $F_2 = aA_2^2A_4e^{i(2\theta_2-\theta_4)} - cA_1A_2A_3e^{i(\theta_1+\theta_2-\theta_3)} + dA_1^2A_4e^{i(2\theta_1-\theta_4)}$.

Let the coefficients of other nonlinear terms except for terms $\phi_1^2(x,t)\phi_1^*(-x,t)$ and $\phi_2^2(x,t)\phi_2^*(-x,t)$ be 0, and the coefficients of $\phi_1^2(x,t)\phi_1^*(-x,t)$ and $\phi_2^2(x,t)\phi_2^*(-x,t)$ terms in Eq. (14a) and Eq. (14b) be equal, then the coupled Eq. (2) can be decoupled into Eq. (5) with the corresponding constraint conditions.

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References

1. J. K. Yang, *Nonlinear Waves in Integrable and Non-integrable Systems* (Society for Industrial and Applied Mathematics, 2010).
2. G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, 2007).
3. M. Abid, C. Huepe, S. Metens, C. Nored, C. T. Phame, L. S. Tuckermand, and M. E. Brachete, "Gross–Pitaevskii dynamics of Bose–Einstein condensates and superfluid turbulence," Fluid Dyn. Res. **33**(5-6), 509–544 (2003).
4. Y. V. Bludov, V. V. Konotop, and N. Akhmediev, "Matter rogue waves," Phys. Rev. A **80**(3), 033610 (2009).
5. M. J. Ablowitz and Z. H. Musslimani, "Integrable Nonlocal Nonlinear Schrödinger Equation," Phys. Rev. Lett. **110**(6), 064105 (2013).
6. A. K. Sarma, M. A. Miri, Z. H. Musslimani, and D. N. Christodoulides, "Continuous and discrete Schrödinger systems with parity-time-symmetric nonlinearities," Phys. Rev. E **89**(5), 052918 (2014).
7. E. Witten, "APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory," Rev. Mod. Phys. **90**(4), 045003 (2018).
8. J. K. Yang, "General N-solitons and their dynamics in several nonlocal nonlinear Schrödinger equations," Phys. Lett. A **383**(4), 328–337 (2019).
9. T. A. Gadzhimuradovand A and M. Agalarov, "Towards a gauge-equivalent magnetic structure of the nonlocal nonlinear Schrdinger equation," Phys. Rev. A **93**(6), 062124 (2016).
10. J. K. Yang, "Physically significant nonlocal nonlinear Schrödinger equation and its soliton solutions," Phys. Rev. E **98**(4), 042202 (2018).
11. B. Yang and J. K. Yang, "Rogue waves in the nonlocal PT -symmetric nonlinear Schrödinger Equation," Lett. Math. Phys. **109**(4), 945–973 (2019).
12. B. Lin, "Singular solutions of the nonlocal nonlinear Schrödinger equation," Eur. Phys. J. Plus **137**(10), 1151 (2022).
13. X. Huang and L. M. Ling, "Soliton solutions for the nonlocal nonlinear Schrödinger Equation," Eur. Phys. J. Plus **131**(5), 148 (2016).

Optics EXPRESS

14. J. Chen, "Generalized Darboux Transformation and Rational Solutions for the Nonlocal Nonlinear Schrödinger Equation with the Self-Induced Parity-Time Symmetric Potential," J. Appl. Math. Phys. **03**(05), 530–536 (2015).
15. W. P. Zhong, Z. Yang, M. Belic, and W. Y. Zhong, "Breather solutions of the nonlocal nonlinear self-focusing Schrödinger equation," Phys. Lett. A **395**, 127228 (2021).
16. J. P. Wu, "A direct reduction approach for a shifted nonlocal nonlinear Schrödinger equation to obtain its N-soliton solution," Nonlinear Dyn. **108**(4), 4021–4028 (2022).
17. P. Ren and J. G. Rao, "Bright–dark solitons in the space-shifted nonlocal coupled nonlinear Schrödinger equation," Nonlinear Dyn. **108**(3), 2461–2470 (2022).
18. P.S. Vinayagam, R. Radha, U. Al Khawaja, and L. M. Ling, "Collisional dynamics of solitons in the coupled PT symmetric nonlocal nonlinear Schrödinger equations," Commun. Nonlinear Sci. Numer. Simul. **52**, 1–10 (2017).
19. P. S. Vinayagama, R. Radha, U. A. Khawaja, and L. M. Ling, "New classes of solutions in the coupled PT symmetric nonlocal nonlinear Schrödinger Equations with four wave mixing," Commun. Nonlinear. Sci. Numer. Simul. **59**, 387–395 (2018).
20. R. B. Mareeswaran and T. Kanna, "Superposed nonlinear waves in coherently coupled Bose–Einstein condensates," Phys. Lett. A **380**(40), 3244–3252 (2016).
21. H. P. Jia, B Li, R. C. Yang, and J. P. Tian, "Diverse composite waves in coherently coupled inhomogeneous fiber systems with external potentials," Nonlinear. Dyn. **99**(4), 2987–2999 (2020).
22. J. Hansryd, P. A. Andrekson, M. Westlund, J. Li, and P. Hedekvist, "Fiber-based optical parametric amplifiers and their applications," IEEE J. Select. Topics Quantum Electron. **8**(3), 506–520 (2002).
23. Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, 2007).
24. B. Yang and J. K. Yang, "Transformations between Nonlocal and Local Integrable Equations," Stud. Appl. Math. **140**(2), 178–201 (2018).
25. W.R. Sun, B. Tian, X. Y. Xie, J. Chai, and Y. Jiang, "High-order rogue waves of the coupled nonlinear Schrödinger equations with negative coherent coupling in an isotropic medium," Commun. Nonlinear Sci. Numer. Simul. **39**, 538–544 (2016).
26. T. Kanna, M. Vijayajayanthi, and M. Lakshmanan, "Coherently coupled bright optical solitons and their collisions," J. Phys. A: Math. Theor. **43**(43), 434018 (2010).
27. X. Lü and B. Tian, "Vector bright soliton behaviors associated with negative coherent coupling," Phys. Rev. E **85**(2), 026117 (2012).
28. S. S. Chen, B Tian, Y. Sun, and C. R. Zhang, "Generalized Darboux Transformations, Rogue Waves, and Modulation Instability for the Coherently Coupled NSE Nonlinear Schrödinger Equations in Nonlinear," Ann. Phys-Berlin **531**(8), 1900011 (2019).
29. X. Wang and J. Wei, "Three types of Darboux transformation and general soliton solutions for the space-shifted nonlocal PT symmetric nonlinear Schrödinger equation," Appl. Math. Lett. **130**, 107998 (2022).
30. B. Yang and J. K. Yang, "On general rogue waves in the parity-time-symmetric nonlinear Schrödinger equation," J. Math. Anal. Appl. **487**(2), 124023 (2020).
31. J. Yang, H. F. Song, M. S. Fang, and L. Y. Ma, "Solitons and rogue wave solutions of focusing and defocusing space shifted nonlocal nonlinear Schrödinger equation," Nonlinear Dyn. **107**(4), 3767–3777 (2022).
32. M. Li and T. Xu, "Dark and antidark soliton interactions in the nonlocal nonlinear Schrödinger Equation with the self-induced parity-time-symmetric potential," Phys. Rev. E **91**(3), 033202 (2015).